La Géométrie est une science de l’expérience (Michel Cassé).

Do we grant enough consideration to the equilibrium that has kept cathedrals standing for centuries? Ceremony or meeting organizers seem confident when gathering thousands of people under these vaults spanning 12 meters or more at a height of 30 or 40 meters, which stand thanks to the mutual thrust hundreds of stone blocks exert on each other. Do we understand through which mental processes those monumental monsters acquired such a standing and lasting privilege? Do we realize that the stability of such a heavy stony sky stems from an exact choice of the right geometrical shape?

The question is solved before even being asked, since—as we all know—technology is a privilege of present modern times. No science before Newton, neither before Galileo . . . Thus uttering that: «the cathedral’s roots lie in empiricism, or the cathedral results from a series of experiments» is to be regarded as a poor answer. Can we imagine people pursuing the best use and effectiveness of their means, at the same time expecting unlikely benefits from random casualties?

The term «the Middle Ages» suggests a mere and mediocre intermission. That a medieval science could have existed in the Middle Ages in Western Europe is a possibility widely underevaluated in what is commonly pictured of Europe’s 12th and 13th centuries. On the contrary, here, we do assume that «the Middle Ages» found its Mechanics in its Geometry. We suggest that its Mathematics lies in the substance of its major architectural works.

In Byzantium and Baghdad, memories of Alexandria’s scientific, technical and intellectual knowledge were carefully kept. No impenetrable border existed in these times moved by antagonisms and covetousness. Trading dealt not only with goods, but also with pictures and ideas. Amalfi and Salerno, Pisa, Genova and Venice collected messages from overseas. Close by, al-Andaluz —Cordoba, Toledo, Zaragoza— gave access to the Omeyad and Abbassid cultures full of antique thoughts (greek, indian, mesopotamian). Palermo was a crossway of cultures. Expeditions of Frankish people on their way through Constantinople to the Holly Land discovered models and arguments they promptly reinvested in material organization, in military and building activities. Fast technical percolations through the whole Mediterranean Sea remind us that mental communication grows in proportion to the material difficulty of communication.

In Western Europe fed with a utilitarian and mixed tradition, the desire of exploring pushed some individuals on the roads, watched closely by more or less open-minded people among their fellow citizens, or on the contrary entrusted by them.
AN ARCHITECTURE OF «ARCEAUX»- THE RIGHT DESIGN FOR AN ARCH

In French crosswords: «plus fort s’il est brisé».

Bits of receipts and of arguments of medieval builders still remain: Hontanon, Aranda, Derand, Belidor, Blondel, Viollet-le-Duc . . . state a rule: chords underlying three parts cut in a segment of a circle extending between two supports, have determined length and pit: their length duplicated downward along their own axis indicates the right thickness to be given to the bearing walls . . . Absurd says Belidor, since neither weight and thickness of the arch nor how high the wall could be, have been taken into account! But when considering the pit of the two chords, looking at them as a representation of the tangents at the springings of the arch—if supposed to be a catenary arch!—that pit exhibits the ratio of the weight of the arch to the horizontal thrust of it . . .

Crucial information: I know the weight—I raised it by my own hands—so I know the horizontal thrust. A strange curve the Catenary: to assume that the shape of the hanging chain stems to its minimal potential energy state, defines that curve completely.
A being of the mechanical world also belongs to the geometrical world. So appears an objective link of powerful meaning between Mechanics and Geometry. In fact, with all the precision we want, we can imitate that «optimal curve» by assembling a series of short circular arches of varied curvatures. The enlightened Builder will mime the Catenary with the gestures he knows and receipts given by the Geometer: by using circles drawn from different centers (Fig. 2).

First of all the Builder had to accept that the specific curvature marking the equilibrium of a hanging chain also marks the equilibrium of the heavy arch in tension between two fixed supporting points. He had to see that the envelope prepared by the stone cutter when enclosing a certain amount of matter inside a pair of circles, should host that equilibrium curve in a comfortable way. He had to understand that should that curve hardly escape an excessively thin enveloppe, the latter would dramatically fold at the spot of the undesirable excursion... and the arch collapse. To seize the tie linking an horizontal force and a vertical one by drawing an oblique line gives new power to geometric approaches. The Catenary works like a horizontal to vertical force converter (Fig. 3). With an obstination worth of compass's one when looking toward the North —rope and chain freely hanging show under which geometric necessities Nature grants them equilibrium. The mechanical algorithm provided by the mere handling of very common objects of the time (handling ropes in naval and military technologies for instance) can bring us to an efficient theoretical understanding level, by a rigorous although common reflection. Apprehending what such a logical tie means in terms of recognition of the laws of Nature, could possibly fit with what could run in European brains in the 13th, 12th or 11th...
centuries. This gave birth to a vocabulary specific of the time’s characteristic shapes: arches drawn on the basis of several centers, among which mainly, the pointed arches. So the specific quality of arches shaped in that way looks clear: it is, more than a supposed minor horizontal thrust, an improved stability, since—despite less material—the pointed shape prevents from the risk of flexion which is a deadly danger for arches.

The specific shape of any pointed arch can be specified in an objective way by a numeral index. For instance the term \( \frac{1}{\cos \theta} \), where \( \theta \) represents the angle of the radius reaching the key of the arch related to the horizon. Since we can find sketches of this in some pages of the famous Villard de Honnecourt Album, and as an homage to him, we suggest to indicate it by the letter H. We can also use the angle \( \theta \) itself, or the position of the centers of the circles on the diameter of the pointed arch. It was referred, in didactic transmissions, to various families corresponding to steps regularly inscribed along those scales. For instance the so-called «arc en tiers point», although the exact meaning of that term is now being discussed.

Every structural functions required determined shapes, so in a given monument—depending on its own cultural context—we find a specific range of arches shapes (Fig. 4).

How about a yet-to-be-invented theory for big masonry structures, which would lay the rigourousness of a geometric determination as a master principle? It seems that a pragmatic reflection, chewed over and over and carefully tested—far from any kind of calculation in the sense we give to the word today—indicated the way: handling ropes in front of circles drawn on the surface of a testing wall, conveys the same indications as the graphic calculations of experiments led on the computer (i.e.: arch’s thickness, level for abutments . . . ) (Fig. 5). Most probably that theory had been first developed in the East, where, long ago—before the year One Thousand AD—and for specific reasons, most arches differed from the mere circle (Byzantium, Sassanid Empire, Egypt). Architectures grew from such shapes, which could not let unaware European travellers eager to foster their own skills and performances at the dawn of a new rationalist age.

Should such an interpretation be accepted, we should also consider that an audacious challenge—the invention of the cathedrals—has possibly been achieved without using anything similar to the analytical algorithms which flourished with a growing refinement in the writings of Galilée, Descartes, Newton, Leibniz, Lagrange, Mach, etc. Success may have been the result of a skilled use of Geometry, and if we do not find explicitly related clues and demonstrations in ancient literate—or if we are not prepared to read them—such a knowledge is peremptorily exhibited in the cathedrals themselves: aerial pavings forming shells 25 or 30 cm thick over 12 meter span—23 meters at Gerona—inscribe these science and understanding in space and time.

«Ronds Points» Sanctuaries in Circle: the Age of anthropic Centrality

A particular «focus» in the large cathedral church in the making was the Sanctuary, an object of distant contemplation for the people, an area accessible to the actors of the cult only. It is the place for the closest approach to the sacred species; i.e. for eucharisty and relics. The sanctuary is a safe room dedicated to fervour and contemplation, the focus of a grandiose staging proper to capturing the attention of crowds. A place where a sensitive and rigorous achievement in geometry had to be elaborated.

In a great number of churches it is by no way possible to walk around the sanctuary. It is so, at a small scale, in most of the monastic cistercian churches, and at a greater scale in carolingian cathedrals and also in a monument such as the Spire cathedral. Similarly at the hugest possible dimension in Haggia Sophia, the sanctuary backs a wall pierced with flows of light, contrasting with the luxurious galleries overhanging the central room. (Fig. 6 a, b, c, d)

On the contrary, at the beginning of the 12th century in the Loire Valley, a couple of monastic churches appear strongly articulated in an externally massive and internally hollow pyramidal shape. The design here is completely different. The idea of winding in concentric rings around the focal point, a row of huge pillars, and a deambulatory gallery, and a series of secondary sanctuaries—the apsidal chapels, looking like epicycles . . . —will be brought to emphasis up to become the specific mark of French architecture in the 12th and 13th centuries. This had been introduced
The geometer and the cathedral

Figure 4
Arcograms
Figure 5

H = 3.767
k = 1.355
and stimulated as pilgrimages to relics increased at the end of the 11th century. The case of the crypta at the cathedral of Clermont-Ferrand is given as an early model (Fig. 7). The generating paradigm consists in a large circle drawn upon the ground, and evenly divided according to precise intentions — structural commitments and symbolic assumptions — as an echo to the Rotunda of the Holly Sepulchre in Jerusalem. The outcoming dial is often structured according to an even division of the circumference — like islamic astronomical devices were. The classic sign of solemnity in the Antiquity — the dramatic shell of the apses — is amplified by echoing rings in peripherical spaces. At the very focus is the front of the altar: the immaterial vertical axis is indirectly but strongly emphasized, as an exaltation of the concept of centrality which will later be outdated by Nicolas de Cues, Copernic and Giordanno Bruno.

All the necessary means, materials and tools were then set to work for promoting a prolific architecture of a kind never seen before. These were: stone and/or terra cotta, wood, iron, lead; the mental ability to decompose a large and complex concept and to extract from it accurately defined geometric components for recomposing the whole; teams of craftsmen producing in a fast and precise way thousand of pieces highly different from one another; perfectly operating transport and lifting devices produced by a technology acquired in military experiments; the activity of small groups of men trained in conceiving and prescribing the huge and unprecedented machinery, as a whole as well as in details; the will of wealthy and mighty lords having an eager desire for these works. This was the time when, in terms of structure design, the Carpenter's strictly linear writing will combine
with the «arches language» which—from then on—will rule the production of vaulting, namely «la croisée d’ogives». Then starts the period we strangely consented to name after the meaningless epithete of «Gothic», that is roughly between 1090 to 1140 (the Sanctuary of Suger at Saint Denis) (Fig. 8).

Acting in the core of all is the stimulating pressure of the culture of geometry. Geometry mind—like a language—is a vector in that world. The way the 12th century’s layouts refined shows how the knowledge acquired in learned spheres—religious and scholar ones, close if not merged into one another—developed. To cast a large radius circle on the ground and cut along its circumference a division whose requirement will rule the whole construction from foundations up to the roofing, requires an effective skill (Fig. 7). We too often look for defects, and neglect to see the strength of the design and of its material projection. The diocesan schools—Adalbéron, Fulbert, Fulbert’s Schools—one century and more before universities, articulated their education around the classical seven arts. Arithmetics, Music, Geometry and Astrology(-nomy) dealt with quantities. Dialectics, Rhetories and Grammar organized reasoning and the expressions of it. In the Portal of the Virgin in Chartres—mid 12th century—Geometry and Arithmetics stand at the top. The «hardcore» of intellectual activities and the crucial role it was entrusted with is not recognized enough, nor the sources it proceeded from. It was freed thanks to the audacious and intelligent initiative of great minds (Gerbert d’Aurillac just before the Year One Thousand is the archetype of such men) who looked started towards muslim countries, searching the marvelous knowledge of Byzance and of the Antiquity collected there.

The general scheme of the radiating Sanctuary is declined differently from a cathedral to another. Variations in dividing the circle in the architectural space reveal an ambition aiming at perfection. Some examples say essentials about a research that was led and renewed over more than a century, since Paray-le-Monial or Fontevrault up to Beauvais. The sample shown here, although brief, refers to major buildings (Fig. 9 a, b, c). A mathematical singularity appears: the division of the circle operated for each of them refers to dividing numbers included in the Fermat series whose current term is expressed as: $2^{2^x} + 1$. Is it an allusion to the conditions of the feasibility in dividing the circle? In modern Europe, the fundamental meaning of the division of the circle was enlightened by Karl Friedrich Gauss in 1796. Whether proceeding algebraically or geometrically, such speculations are submitted to a sole absolute logical link: cyclotomic numbers give the ground of the polynomial equation theory . . . So then?

To conclude

Mathematical options took place in the very substance of the large cathedral, at the early beginning of its conception, not for the sake of cosmetic conveniences but to establish its material framework, and to insure architecture could stand and remain in the world of gravity. A consistant Geometry, accountable for stability throughout the time. Cathedrals open a large field for the action of Geometry: (1) to recognize the
The geometer and the cathedral

Cathédrale de Bourges

Notre-Dame de Paris

Cathédrale de Beauvais

Figure 9
conditions of stable shape, the noblest among the Architect-Geometer's duties, (2) to design the sacred area as a spiritual astrolab, (3) to combine easy and efficient processes for translating the mental scheme into a real scale on the ground.

The first challenges being solved and a culture of mechanic-geometry having been built, the organizing inclination of the Geometer was then called by a number of topics: the field of carving and pictural composition, constituted by an unnumered amount of details up to the miniature architecture in the architecture. Some approaches served mnemonic more than heuristic purposes, such as conserved didactic writings concerning pinnacle's design or net vaulting's pattern (Fig. 10) . . . The law of necessity is not as strong then. Fancy wanders freely. No doubt that professionalism and expert's proudness rapidly produced enclosed, jealously protected islands of knowledge. No doubt that in such an environment - alongside the «Art du Traict» itself — legends flourished about supposed aesoteric items of knowledge, letting essentials fall into oblivion. Without a clear option for reasonable things, we can endlessly search for supposed theoretical drawings in architectures, and probably imagine rather idle chimaeras compared with the major commitments recorded above.

Does a consciousness of Geometry proceed the need for Geometry? Or inversely? There was a Geometry for agricultural people, one for Megaliths, one for Roofs, for Chariots, for Ships, a Geometry for Arches and Vaults Designers, there were Newton's and Leibniz' Geometry of curves and trajectories. Einsteinian Chrono-Geometry of cosmic spaces is not very familiar yet. Now we keep to the Geometry of Monge, Laplace and Eiffel, suitable to industrial forecasting.

We should not forget that in Europe, a thousand years ago, a Geometry and its Geometer appeared, necessary for inventing Cathedrals.